Solving Linear Inequalities

**Solutions to Inequalities**

To solve a linear inequality means to find the values of the variable that satisfy the inequality. Unlike linear equations, there may be many values that satisfy the inequality and these values will be represented by an interval on the number line or using interval notation.

**Example:** Represent the inequality \( x < 2 \) on a number line and using interval notation.

**Solution:** On a number line the solution may be represented as follows. The open circle at 2 represents the fact that 2 is not a solution to the inequality and the arrow to the left represents the fact that all values less than 2 are solutions.

![Open circle on number line]

The solution in interval notation is \((-\infty, 2)\). Because 2 is not a solution we use “paren” and a “paren” is always used with the infinity symbol.

**Example:** Represent the inequality \( x \geq -1 \) on a number line and using interval notation.

**Solution:** The closed circle at -1 represents the fact that -1 is a solution to the inequality. The arrow to the right represents the fact that all solutions greater than -1 are solutions.

![Closed circle on number line]

The solution in interval notation is \([-1, \infty)\). Because -1 is a solution we use a square bracket and as always “paren” is used with the infinity symbol.

**Example:** Represent the inequality \(-1 \leq x < 2\) on a number line and using interval notation.

**Solution:** This is a compound inequality where the values of x are less than 2 but greater than or equal to -1.

![Closed circle and open circle on number line]

The solution in interval notation is \([-1, 2)\).
Solving Linear Inequalities

The procedure for solving a linear inequality is very similar to solving a linear equation as demonstrated in the following examples. There is however one very important difference as you will see.

**Example:** Solve the linear inequality \( 2x - 3 \leq 13 \)

**Solution:**

\[
\begin{align*}
2x - 3 & \leq 13 \\
2x & \leq 16 \\
x & \leq 8 
\end{align*}
\]

Therefore, the solution expressed in interval notation is \((-\infty, 8]\)

**Example:** Solve the linear inequality \( 3x + 1 > -1 + x \)

**Solution:**

\[
\begin{align*}
3x + 1 & > -1 + x \\
2x & > -2 \\
x & > -1 
\end{align*}
\]

Therefore, the solution expressed in interval notation is \((-1, \infty)\)

**Example:** Solve the linear inequality \(-5x + 3 \geq 2 + 3x\)

**Solution:** This example demonstrates the important difference between solving equations and inequalities. When applying the multiplication property of equality to a negative number, you **MUST** change the direction of the inequality symbol.

\[
\begin{align*}
-5x + 18 & \geq 2 + 3x \\
-8x & \geq -16 \\
x & \leq 2 
\end{align*}
\]

Therefore, the solution expressed in interval notation is \((-\infty, 2]\)

**Example:** Solve the linear inequality \(3(x - 4) + 2x < -2(5 - x)\)

**Solution:**

\[
\begin{align*}
3(x - 4) + 2x & < -2(5 - x) \\
3x - 12 + 2x & < -10 + 2x \\
5x - 12 & < -10 + 2x \\
3x & < 2 \\
x & < \frac{2}{3} 
\end{align*}
\]

Therefore, the solution expressed in interval notation is \((-\infty, \frac{2}{3})\)
Example: Solve the compound linear inequality \(12 < 2x - 6 \leq 24\)

Solution: This type of compound inequality is called a conjunction. A conjunction is usually written as a single statement with three parts; left, middle, and right. A conjunction can also be written as two separate statements separated with the word “and” such as

\[
12 < 2x - 6 \quad \text{and} \quad 2x - 6 \leq 24
\]

Solving a compound inequality involves using the addition and multiplication property of equality on all three parts of the inequality.

\[
\begin{align*}
12 < 2x - 6 & \leq 24 \\
18 < 2x & \leq 30 \\
9 < x & \leq 15
\end{align*}
\]

Add 6 to all three parts.
Divide all three parts by 2

Therefore, the solution expressed in interval notation is \((9, 15]\)

Example: Solve the compound linear inequality \(-2 \leq -4x - 1 < 5\)

Solution:

\[
\begin{align*}
-2 & \leq -4x - 1 < 5 \\
-1 & \leq -4x < 6 \\
\frac{1}{4} & \geq x > -\frac{3}{2} \\
-\frac{3}{2} & < x \leq \frac{1}{4}
\end{align*}
\]

Add 1 to all parts
Divide all parts by -4 (change direction of signs)
Reverse Inequality

Therefore, the solution expressed in interval notation is \((-\frac{3}{2}, \frac{1}{4}]\)

Example: Solve the compound inequality \(5x - 2 < 3 \quad \text{or} \quad 2x + 1 > 13\)

Solution: This type of compound inequality is called a disjunction. A disjunction will always be written with the word “or” between the two inequality statements. To solve a disjunction, just solve each statement and combine the answers.

\[
\begin{align*}
5x - 2 & < 3 \\
5x & < 5 \\
x & < 1 \\
2x + 1 & > 13 \\
2x & > 12 \\
x & > 6
\end{align*}
\]

Therefore, the solution expressed in interval notation is \((-\infty, 1) \quad \text{or} \quad (6, \infty)\).
Applications:

**Example:** A student has exam scores of 68%, 75%, and 79% in a math course. What must he score on the last exam to earn a B (80% or better) in the course?

**Solution:**

Step 1: Analyze the Problem:
This is an inequality that involves an average as well. Let x = the score on the last exam.

Step 2: Create an Equation:
The equation will be based on the average formula
\[
\frac{68 + 75 + 79 + x}{4} \geq 80
\]

Step 3: Solve the Equation:
\[
\frac{68 + 75 + 79 + x}{4} \geq 80
\]
\[
222 + x \geq 320
\]
\[
x \geq 98
\]

Step 4: State the Conclusion:
The student needs to score at least a 98% on the last exam.

**Example:** A car manufacturer produces three models in equal quantities. One model has an economy rating of 17 miles per gallon, and the second model is rated for 19 mpg. If government regulations require the manufacturer to have a fleet average that exceeds 21 mpg, what economy rating is required for the third model?

**Solution:**

Step 1: Analyze the Problem:
This is an inequality that involves an average as well. Let x = the economy rating on the 3\textsuperscript{rd} model.

Step 2: Create an Equation:
\[
\frac{17 + 19 + x}{3} \geq 21
\]

Step 3: Solve the Equation:
\[
\frac{17 + 19 + x}{3} \geq 21
\]
\[
36 + x \geq 63
\]
\[
x \geq 27
\]

Step 4: State the Conclusion:
The economy rating on the third model must be at least 27 mpg.
**Example:** A restaurant is building a new rectangular counter which must be 5 feet longer than it is wide. Designers have determined that the perimeter needs to exceed 30 feet. Determine the acceptable values for the width and length.

**Solution:**

Step 1: Analyze the Problem:
This is a geometric inequality that will be based on the perimeter formula. Let \( x \) = the width of the counter.

Step 2: Create an Equation:
The perimeter of a rectangle is \( P = 2L + 2W \). Since the perimeter must exceed 30 feet we can create the following inequality.

\[
2(x + 5) + 2x \geq 30
\]

Step 3: Solve the Equation:

\[
\begin{align*}
2(x + 5) + 2x & \geq 30 \\
2x + 10 + 2x & \geq 30 \\
4x & \geq 20 \\
x & \geq 5
\end{align*}
\]

Step 4: State the Conclusion:
The width must be at least 5 feet long and the length must be at least 10 feet long.

**Example:** What numbers satisfy the condition that four more than three times a number is at most 10?

**Solution:**

Step 1: Analyze the Problem:
Let \( x \) = the number.

Step 2: Create an Equation:
The term “at most” implies less than or equal to.

\[
3x + 4 \leq 10
\]

Step 3: Solve the Equation:

\[
\begin{align*}
3x + 4 & \leq 10 \\
3x & \leq 6 \\
x & \leq 2
\end{align*}
\]

Step 4: State the Conclusion:
The number must be no greater than 2.
Example: It costs a student $18.00 to rent a cap and gown and $0.80 for each graduation announcement that she orders. If she does not want her spending to exceed $50.00, how many announcements can she order?

Solution:
Step 1: Analyze the Problem:
Let $x$ = the number of announcements

Step 2: Create an Equation:
The sum of all costs must not exceed $50.00. Therefore, we can create the inequality:

$$0.8x + 18 \leq 50$$

Step 3: Solve the Equation:

$$0.8x + 18 \leq 50$$
$$0.8x \leq 32$$
$$x \leq 40$$

Step 4: State the Conclusion:
The number of announcements must be no greater than 40.

Example: A cellular telephone company has currently enrolled 36,000 customers in a new calling plan. If an average of 1200 people are signing up for the plan each day, in how many days will the company surpass their goal of 150,000 customers enrolled?

Solution:
Step 1: Analyze the Problem:
Let $x$ = number of days to reach its goal.

Step 2: Create an Equation:
We are looking for the number of days when the sum of all customers is greater than 150,000. Therefore,

$$1200x + 36,000 > 150,000$$

Step 3: Solve the Equation:

$$1200x + 36,000 > 150,000$$
$$1200x > 114,000$$
$$x > 95$$

Step 4: State the Conclusion:
The company will reach its goal after 95 days.
Example: To hold the temperature of a room between 18 and 27 degrees Celsius, what Fahrenheit temperatures must be maintained? Use the formula $C = \frac{9}{5}(F - 32)$

Solution:
Step 1: Analyze the Problem:
Because we are not told whether to include the endpoints, I will include.

Step 2: Create an Equation:
$$18 \leq \frac{9}{5}(F - 32) \leq 27$$

Step 3: Solve the Equation:
$$\frac{5}{9}(18) \leq \frac{5}{9} \cdot \frac{9}{5}(F - 32) \leq \frac{5}{9}(27)$$
$$10 \leq F - 32 \leq 15$$
$$42 \leq F \leq 47$$

Step 4: State the Conclusion:
The temperature of the room must be held between 42 and 47 degrees Fahrenheit.

Example: Taylor is shopping for a motorcycle in Peoria where the sales tax is 8.15% and the title and license fee is $250.00. If the maximum he can spend is $6,000, then he should look for a motorcycle in what price range?

Solution: Let $x = \text{the price of the motorcycle}$.

Then we may create the following inequality which is based on the total cost being less than or equal to the price of the motorcycle, plus the tax, title and license fee:

$$x + 0.0815x + 250 \leq 6,000$$

Solve the inequality for $x$.

$$x + 0.0815x + 250 \leq 6,000$$
$$1.0815x \leq 5,750$$
$$x \leq 5316.69$$

The price of the motorcycle must be less than $5,316.69$. 
**Example:** Ronald wants to sell his car through a broker who charges a commission of 10% of the selling price. Ronald still owes $11,025 on the car. Ronald must get enough to at least pay off the loan. What is the range of the selling price?

**Solution:** This problem can be written as an IS-OF statement. If the commission is 10%, of the selling price, then the amount Ronald must receive ($11,025) will be 90% of the selling price.

Let \( x \) = the selling price. Then,

\[
0.90x \geq 11,025
\]

\[
x \geq \frac{11,025}{0.9}
\]

\[
x \geq 12,250
\]

Ronald must sell the car for at least $12,250 to pay off his loan.

**Example:** Professor Williamson counts his midterm as 2/3 of the grade and his final as 1/3 of the grade. Wendy scored only 48 on the midterm. What range of scores on the final exam would put Wendy’s average between 70 and 79 inclusive?

**Solution:** Wendy’s grade may be represented by the expression:

\[
48 \left( \frac{2}{3} \right) + x \left( \frac{1}{3} \right)
\]

The inequality then becomes:

\[
70 \leq 48 \left( \frac{2}{3} \right) + x \left( \frac{1}{3} \right) \leq 79
\]

\[
70 \leq 32 + \frac{1}{3} x \leq 79
\]

\[
38 \leq \frac{1}{3} x \leq 47
\]

\[
114 \leq x \leq 147
\]

Wendy must score between a 114 and a 147 on the final exam.
Example: The price of Jill’s favorite big salad at the corner restaurant is 10 cents more than the price of Jerry’s hamburger. After treating a group of friends to lunch, Jerry is certain that for 10 hamburgers and 5 salads he spent more than $9.14, but not more than $13.19. In what price range is a hamburger?

Solution: Start by defining the variable.

Let $x =$ price of hamburger

Then $x + 0.10 =$ the price of a big salad

The inequality is then,

$$9.14 < 10x + 5(x + 0.10) \leq 13.19$$

This is a compound inequality or a conjunction. Solve the inequality to determine the price range of a hamburger.

$$9.14 < 10x + 5(x + 0.10) \leq 13.19$$
$$9.14 < 10x + 5x + 0.5 \leq 13.19$$
$$9.14 < 15x + 0.5 \leq 13.19$$
$$8.64 < 15x \leq 12.69$$
$$0.58 < x \leq 0.85$$

A hamburger is between $0.58 and $0.85. Obviously, this problem is from about 1957.

Example: Tyler scored 65 on his calculus midterm. If his final exam counts twice as much as his midterm exam, then for what range of scores on his final would Tyler get an average between 79 and 90?

Solution: Start by defining the variable.

Let $x =$ grade on final exam

Since this is an averaging problem we will set it up as such. Since the final counts as twice the midterm we need to multiply it by 2. The denominator is three since we are essentially adding up three scores (the final twice).

$$79 \leq \frac{65 + 2x}{3} \leq 90$$

Solve this compound inequality to determine the range of possible scores on the final.

$$79 \leq \frac{65 + 2x}{3} \leq 90$$
$$237 \leq 65 + 2x \leq 270$$
$$172 \leq 2x \leq 205$$
$$86 \leq x \leq 102$$

Tyler must score between an 86 and a 102 on the final.
Example: United Parcel Service defines girth as the sum of the length, twice the width, and twice the height. The maximum length that can be shipped with UPS is 108 in. and the maximum girth is 130 in. If a box has a length of 40 in. and a width of 30 in. then in what range must the height fall?

Solution: The girth may be defined algebraically as follows.

\[ G = L + 2W + 2H \]

The girth must be less than or equal to 130. Substitute in this fact along with the length and width of the box.

\[
\begin{align*}
L + 2W + 2H & \leq 130 \\
40 + 60 + 2H & \leq 130 \\
100 + 2H & \leq 130
\end{align*}
\]

Now, solve this inequality for the height.

\[
\begin{align*}
100 + 2H & \leq 130 \\
2H & \leq 130 - 100 \\
2H & \leq 30 \\
H & \leq 15
\end{align*}
\]

The height must be less than or equal to 15 in.