Approximating the Area under a Curve

Now that we have established the theoretical development for finding the area under a curve, let’s start developing a procedure to find an actual value for the area. Since we are using rectangles to approximate the area, we will need to find the width of each rectangle and the height of each rectangle. Then we will use the formula

\[ A = H \times W \]

\[ A = f(x)\Delta x \]

to find the area of each rectangle. Finally, we will add up the areas of all the rectangles. One final step will then be introduced and this is where calculus finally enters the picture.

Finding the Width of Each Rectangle under a Curve

When finding the area under the curve of a function we will first divide the interval into a number of rectangles. Each rectangle will have the same width and will be given by the formula:

\[ \Delta x = \frac{b - a}{n} \]

where the interval is \([a, b]\) and \(n\) represents the number of rectangles.

EXAMPLE 1: Find the width of 4 rectangles under a curve over the interval \([1, 9]\).

SOLUTION:

\[ \Delta x = \frac{9 - 1}{4} = \frac{8}{4} = 2 \]

The width of each rectangle is 2.

Determining the Height of Each Rectangle

The height of each rectangle is the function value where the rectangle intersects the curve. Although the rectangle may theoretically intersect the curve anywhere along its top edge, it is convenient to create the rectangles so that they intersect at the left endpoint (LEP), midpoint (MP), or right endpoint (REP) of each rectangle. These points have formulas that we can use to easily calculate the \(x\)-value at which each rectangle will intersect the curve. Once we have determined the \(x\)-value where each rectangle intersects the curve, we will use these points to determine the height of each rectangle. We do this by evaluating the function at each of these points.

LEP \( x = a + \Delta x(k - 1) \) then the height is \( h = f(x) = f(a + \Delta x(k - 1)) \)

REP \( x = a + \Delta xk \) then the height is \( h = f(x) = f(a + \Delta xk) \)

MP \( x = a + \Delta x \frac{(k + k - 1)}{2} \) then the height is \( h = f(x) = f(a + \Delta x \frac{(k + k - 1)}{2}) \)
**EXAMPLE 2:** Find the left endpoint and height of each of 4 circumscribed rectangles over the interval \([0, 1]\) for the function \(f(x) = 1 - x^2\).

**SOLUTION:** First we need to determine the width of each rectangle.

\[
\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} = 0.25 
\]

Then, using the LEP, we have \(x = a + \Delta x(k-1) = 0 + 0.25(k-1) = 0.25(k-1)\), which results in:

Rectangle 1, \(k=1\), \(x = 0.25(1-1) = 0\)
then the height is \(f(0) = 1 - (0)^2 = 1\)

Rectangle 2, \(k=2\), \(x = 0.25(2-1) = 0.25\)
then the height is \(f(0.25) = 1 - (0.25)^2 = \frac{15}{16}\)

Rectangle 3, \(k=3\), \(x = 0.25(3-1) = 0.5\)
then the height is \(f(0.5) = 1 - (0.5)^2 = \frac{3}{4}\)

Rectangle 4, \(k=4\), \(x = 0.25(4-1) = 0.75\)
then the height is \(f(0.75) = 1 - (0.75)^2 = \frac{7}{16}\)

**EXAMPLE 3:** Find the right endpoint and height of each of 4 inscribed rectangles over the interval \([0, 1]\) for the function \(f(x) = 1 - x^2\).

**SOLUTION:** First we need to determine the width of each rectangle.

\[
\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} = 0.25 
\]

Then, using the REP, we have \(x = a + \Delta x k = 0 + 0.25k = 0.25k\)

Rectangle 1, \(k=1\), \(x = 0.25(1) = 0.25\)
then the height is \(f(0.25) = 1 - (0.25)^2 = \frac{15}{16}\)

Rectangle 2, \(k=2\), \(x = 0.25(2) = 0.5\)
then the height is \(f(0.5) = 1 - (0.5)^2 = \frac{3}{4}\)

Rectangle 3, \(k=3\), \(x = 0.25(3) = 0.75\)
then the height is \(f(0.75) = 1 - (0.75)^2 = \frac{7}{16}\)

Rectangle 4, \(k=4\), \(x = 0.25(4) = 0.25\)
then the height is \(f(1) = 1 - (1)^2 = 0\)
EXAMPLE 4: Find the midpoint and height of each of 4 inscribed rectangles over the interval [0, 1] for the function \( f(x) = 1 - x^2 \).

SOLUTION: First we need to determine the width of each rectangle.

\[
\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} = 0.25
\]

Then, using the MP, we have

\[
x = a + \Delta x \left( \frac{k+1}{2} \right) = 0 + 0.25 \left( \frac{2k-1}{2} \right) = \frac{2k-1}{8}
\]

Rectangle 1, \( k=1 \),

\[
x = \frac{2(1)-1}{8} = \frac{1}{8} = 0.125
\]

then the height is

\[
f(0.125) = 1 - (0.125)^2 = \frac{63}{64}
\]

Rectangle 2, \( k=2 \),

\[
x = \frac{2(2)-1}{8} = \frac{3}{8} = 0.375
\]

then the height is

\[
f(0.375) = 1 - (0.375)^2 = \frac{55}{64}
\]

Rectangle 3, \( k=3 \),

\[
x = \frac{2(3)-1}{8} = \frac{5}{8} = 0.625
\]

then the height is

\[
f(0.625) = 1 - (0.625)^2 = \frac{39}{64}
\]

Rectangle 4, \( k=4 \),

\[
x = \frac{2(4)-1}{8} = \frac{7}{8} = 0.875
\]

then the height is

\[
f(0.875) = 1 - (0.875)^2 = \frac{15}{64}
\]

Now that we have seen how to find the height of each rectangle using the left endpoint, right endpoint and midpoint, it should be obvious that the REP provides the easiest calculations because the formula \( x = a + \Delta x \) is simpler than the others. The following example uses the right endpoint formula.

EXAMPLE 5: Find the height of each of 4 rectangles under the curve of the function \( f(x) = x^2 + 2 \) over the interval [2, 14] using the REP to determine the height of each rectangle.

SOLUTION: Determine the width of each rectangle.

\[
\Delta x = \frac{b-a}{n} = \frac{14-2}{4} = \frac{12}{4} = 3
\]

Using the REP formula, we have \( a + \Delta x k = 2 + 3k \), which results in:

Rectangle 1, \( k=1 \), \( x = 2 + 3(1) = 5 \) the height is \( h = f(5) = (5)^2 + 2 = 25 + 2 = 27 \)

Rectangle 2, \( k=2 \), \( x = 2 + 3(2) = 8 \) the height is \( h = f(8) = (8)^2 + 2 = 64 + 2 = 66 \)

Rectangle 3, \( k=3 \), \( x = 2 + 3(3) = 11 \) the height is \( h = f(11) = (11)^2 + 2 = 121 + 2 = 123 \)

Rectangle 4, \( k=4 \), \( x = 2 + 3(4) = 14 \) the height is \( h = f(14) = (14)^2 + 2 = 196 + 2 = 198 \)

The heights of each of the four rectangles are 27, 66, 123, and 171.
Total Area under the Curve:

Now that we know how to find the width and height of each rectangle we are ready to find the area of each rectangle and the total area of the rectangles under the curve. To do this we use the formula $A = HW$, where $H$ is the height of each rectangle, and $W$ is the width of each rectangle. This formula can be modified using function notation to $A = f(x^*)\Delta x$ where $x^*$ represents the LEP, REP or MP of each rectangle.

The basic procedure for finding the area under a curve is:

1. Calculate the width of each rectangle $\Delta x$.
2. Find the height of each rectangle $f(x)$.
3. Multiply $f(x)\Delta x$ to find the area of each rectangle.
4. Add all the areas together to find the total area $A_T$.

**EXAMPLE 6:** Find the area under the curve of the function $f(x) = x^2 + 1$ over the interval $[1, 9]$ using the REP formula and four rectangles.

**SOLUTION:**

**Step 1:** Determine the width of each rectangle.

$$\Delta x = \frac{b-a}{n} = \frac{9-1}{4} = \frac{8}{4} = 2$$

**Step 2:** Using the REP formula, we have $a + \Delta xk = 1 + 2k$, which results in:

- Rectangle 1, $k=1$, $x = 1 + 2(1) = 3$ the height is $h = f(3) = (3)^2 + 1 = 10$
- Rectangle 2, $k=2$, $x = 1 + 2(2) = 5$ the height is $h = f(5) = (5)^2 + 1 = 26$
- Rectangle 3, $k=3$, $x = 1 + 2(3) = 7$ the height is $h = f(7) = (7)^2 + 2 = 51$
- Rectangle 4, $k=4$, $x = 1 + 2(4) = 9$ the height is $h = f(9) = (9)^2 + 2 = 83$

**Step 3:** Multiply $f(x)\Delta x$ to find the area of each rectangle.

- $A_1 = f(x)\Delta x = f(3)(2) = (10)(2) = 20$
- $A_2 = f(x)\Delta x = f(5)(2) = (26)(2) = 52$
- $A_3 = f(x)\Delta x = f(7)(2) = (51)(2) = 102$
- $A_4 = f(x)\Delta x = f(9)(2) = (83)(2) = 166$

**Step 4:** Add all the areas together to find the total area.

$$A_T = A_1 + A_2 + A_3 + A_4 = 20 + 52 + 102 + 166 = 340$$

In each of the above examples, we have used a small number of rectangles. As we increase the number of rectangles, the amount of excess area or shortage of area will decrease. By letting $n$ be a very large number, we will obtain an approximate area that is closer to the actual area under the curve. Before working with large numbers of rectangles we need to acquaint ourselves with the concept of Sigma Notation.