6.3 – Applications of Normal Distributions

Objectives:
1. Find probabilities and percentages from known values.
2. Find values from known areas.

Overview:
This section presents methods for working with normal distributions that are not standard. That is, the mean is not 0 or the standard deviation is not 1, or both. The key concept is that we can use a simple conversion that allows us to standardize any normal distribution so that the methods of the previous lesson can be used.

Conversion Formula:
In a previous lesson, we learned how to convert data values into z-scores. We will once again use this formula to work with non-standard normal distributions.

\[ z = \frac{x - \mu}{\sigma} \]

Converting to a Standard Normal Distribution:
By converting selected data values to z-scores, we may use the procedures learned for determining probabilities using the standard normal distribution.

When finding areas with a nonstandard normal distribution, use this procedure:
1. Sketch a normal curve, label the mean and specific x values, and then shade the region representing the desired probability.
2. For each relevant value x that is a boundary for the shaded region, convert that value to the equivalent z-score.
3. Use table A-2 or Excel to find the area of the shaded region. This area is the desired probability.
Example: The safe load for a water taxi is determined to be 3500 pounds and the mean weight of a passenger is 140 pounds. Assume that all passengers are men. Assume also that the weights of the men are normally distributed with a mean of 172 pounds and standard deviation of 29 pounds. If one man is randomly selected, what is the probability that he weighs less than 174 pounds?

Solution: Use the three steps for finding areas with a nonstandard normal distribution.

Step 1: Sketch normal curve and shade appropriate region.

Step 2: Convert 174 pounds to a z-score.

\[
z = \frac{x - \mu}{\sigma} = \frac{174 - 172}{29} = 0.007
\]

Step 3: Using the z-score \( z = 0.007 \) and table A-2, we determine the area to the left of this score to be 0.5279.

Therefore, the probability that a male passenger weighs less than 174 pounds is 0.5279.

Example: The mean is \( \mu = 60.0 \) and the standard deviation is \( \sigma = 4.0 \). Find the probability that \( X \) is less than 53.0.

Solution: Use the three steps for finding areas with a nonstandard normal distribution.

Step 1: Sketch normal curve and shade appropriate region.

Step 2: Convert 53 to a z-score.

\[
z = \frac{x - \mu}{\sigma} = \frac{53 - 60}{4} = -1.750
\]

Step 3: Using the z-score \( z = -1.750 \) and table A-2, we determine the area to the left of this score to be 0.0401.

Therefore, the probability that \( x \) is less than 53 is 0.0401.
Example: IQ scores of adults are normally distributed with a mean of 100 and a standard deviation of 15 (as on the Wechsler test). Find the probability that a randomly selected adult will have an IQ between 85 and 125.

Solution: Use the three steps for finding areas with a nonstandard normal distribution.

Step 1: Sketch normal curve and shade appropriate region.

Step 2: Convert 85 and 125 to z-scores.

\[
\begin{align*}
    z &= \frac{x - \mu}{\sigma} = \frac{85 - 100}{15} = -1.00 \\
    z &= \frac{x - \mu}{\sigma} = \frac{125 - 100}{15} = 1.667
\end{align*}
\]

Step 3: Using the z-scores \( z = -1.00 \), \( z = 1.667 \) and table A-2, we determine the area of the shaded region to be

\[
\text{Area} = 0.9515 - 0.1587 = 0.7938
\]

Therefore, the probability that a randomly selected adult has an IQ between 85 and 110 is 0.7938.

Example: The volumes of soda in quart soda bottles are normally distributed with a mean of 32.3 oz and a standard deviation of 1.2 oz. What is the probability that the volume of soda in a randomly selected bottle will be less than 32 oz?

Solution: Use the three steps for finding areas with a nonstandard normal distribution.

Step 1: Sketch normal curve and shade appropriate region.

Step 2: Convert 32 to a z-score.

\[
    z = \frac{x - \mu}{\sigma} = \frac{32 - 32.3}{1.2} = -0.250
\]

Step 3: Using the z-score \( z = -0.250 \) and table A-2, we determine the area to the left of this score to be 0.4013.

Therefore, the probability that the volume of soda in a randomly selected bottle will be less than 32 oz is 0.0401.
Example: The weekly salaries of teachers in one state are normally distributed with a mean of $490 and a standard deviation of $45. What is the probability that a randomly selected teacher earns more than $525 a week?

Solution: Use the three steps for finding areas with a nonstandard normal distribution.

Step 1: Sketch normal curve and shade appropriate region.

Step 2: Convert 525 to a z-score.

\[ z = \frac{x - \mu}{\sigma} = \frac{525 - 490}{45} = 0.783 \]

Step 3: Using the z-score \( z = 0.783 \) and table A-2, we determine the area to the left of this score to be 0.7823. Because the desired area is to the right, we will subtract 1 to obtain 0.217

Therefore, the probability the probability that a randomly selected teacher earns more than $525 a week is 0.0217.

Example: A bank's loan officer rates applicants for credit. The ratings are normally distributed with a mean of 200 and a standard deviation of 50. If an applicant is randomly selected, find the probability of a rating that is between 170 and 220.

Solution: Use the three steps for finding areas with a nonstandard normal distribution.

Step 1: Sketch normal curve and shade appropriate region.

Step 2: Convert 170 and 220 to z-scores.

\[ z = \frac{x - \mu}{\sigma} = \frac{170 - 200}{50} = -0.600 \]
\[ z = \frac{x - \mu}{\sigma} = \frac{220 - 200}{50} = 0.400 \]

Step 3: Using the z-scores \( z = -0.600, z = 0.400 \) and table A-2, we determine the area of the shaded region to be

\[ \text{Area} = 0.6554 - 0.2743 = 0.381 \]

Therefore, the probability that a rating is between 170 and 220 is 0.381.
Finding Values from Known Areas:
In addition to finding areas or probabilities given a value, we can also find a value from a given probability or area.

Procedure for Finding Values Using Table A-2 and the Formula:
1. Sketch a normal distribution curve, enter the given probability or percentage in the appropriate region of the graph, and identify the $x$ value(s) being sought.
2. Use Table A-2 to find the $z$ score corresponding to the cumulative left area bounded by $x$. Refer to the body of Table A-2 to find the closest area, then identify the corresponding $z$ score.
3. Using Formula 6-2, enter the values for $\mu$, $s$, and the $z$ score found in step 2, then solve for $x$.

$$x = \mu + (z \cdot s)$$

*If $z$ is located to the left of the mean, be sure that it is a negative number.*

4. Refer to the sketch of the curve to verify that the solution makes sense in the context of the graph and the context of the problem.

Example: The safe load for a water taxi is determined to be 3500 pounds. Assume also that the weights of the all passengers are normally distributed with a mean of 172 pounds and standard deviation of 29 pounds. Determine what weight separates the lightest 99.5% from the heaviest 0.5%?

Solution:
Sketch a normal distribution curve and identify the $x$ value(s) being sought.

From table A-2 the $z$-score corresponding to an area of 0.9950 is 2.575

Solving the formula for $x$.

$$x = \mu + (z \cdot s)$$

$$x = 172 + (2.575 \times 29)$$

$$x = 172 + 74.675$$

$$x = 246.675$$

$$x \approx 247$$

The weight of 247 pounds separates the lightest 99.5% from the heaviest 0.5%
**Example:** The graph depicts IQ scores of adults, and those scores are normally distributed with a mean of 100 and a standard deviation of 15 (as on the Wechsler test). The shaded area under the curve is 0.5675. Find the corresponding IQ score.

![Graph depicting IQ scores](image)

**Solution:**
From table A-2 the z-score corresponding to an area of 0.5675 is 0.17.

Solving the formula for \(x\).

\[
\begin{align*}
  x &= \mu + (z \cdot s) \\
  x &= 100 + (0.17 \times 15) \\
  x &= 100 + 2.550 \\
  x &= 102.550 \\
  x &\approx 103
\end{align*}
\]

The IQ score corresponding to an area of 0.5675 is 103.

**Example:** The serum cholesterol levels for men in one age group are normally distributed with a mean of 178.3 and a standard deviation of 40.4. All units are in mg/100 mL. Find the two levels that separate the top 9% and the bottom 9%.

**Solution:**
From table A-2 the z-scores are as follows:

Bottom 9% z-score = -1.34 (area = 0.09) and Top 9% z-score = 1.34 (area = 0.91)

Solving the formula for \(x\).

\[
\begin{align*}
  x &= \mu + (z \cdot s) \\
  x &= 178.3 + (\cdot(-1.34\times40.4)) \\
  x &= 178.3 - 54.136 \\
  x &= 124.164 \\
  x &\approx 124
\end{align*}
\]

\[
\begin{align*}
  x &= \mu + (z \cdot s) \\
  x &= 178.3 + (\cdot(1.34\times40.4)) \\
  x &= 178.3 + 54.136 \\
  x &= 232.436 \\
  x &\approx 232
\end{align*}
\]

The two levels are 124 mg/100mL and 232mg/100mL.
Example: In one region, the September energy consumption levels for single-family homes are found to be normally distributed with a mean of 1050 kWh and a standard deviation of 218 kWh. Find $P_{45}$, which is the consumption level separating the bottom 45% from the top 55%.

Solution:
From table A-2 the z-score corresponding to an area of 0.4500 is -0.13.

Solving the formula for $x$.

\[ x = \mu + (z \cdot s) \]
\[ x = 1050 + (-0.13 \times 218) \]
\[ x = 1050 - 28.34 \]
\[ x = 1021.66 \]
\[ x \approx 1022 \]

The IQ score corresponding to an area of 0.4500 is 1022

Helpful Hints for Finding Values:
1. Don’t confuse z scores and areas. Z-scores are distances along the horizontal scale, but areas are regions under the normal curve.
2. Table A-2 lists z scores in the left column and across the top row, but areas are found in the body of the table.
3. Choose the correct (right/left) side of the graph.
4. A z score must be negative whenever it is located in the left half of the normal distribution.
5. Areas (or probabilities) are positive or zero values, but they are never negative.
Applications with Normal Distributions

1. Start
2. What do you want to find?
   - Find a probability (from a known value of $x$)
   - Find a value of $x$ (from known probability or area)

3. Are you using technology or Table A-2?
   - Technology
   - Table A-2

4. Convert to the standard normal distribution by finding $z$:
   \[ z = \frac{x - \mu}{\sigma} \]
   - Look up $z$ in Table A-2 and find the cumulative area to the left of $z$.

5. Find the probability by using the technology.
   - Identify the cumulative area to the left of $x$.

6. Are you using technology or Table A-2?
   - Technology
   - Table A-2

7. Find $x$ directly from the technology.
   - Solve for $x$:
     \[ x = \mu + z \cdot \sigma \]
   - Look up the cumulative left area in Table A-2 and find the corresponding $z$ score.