9.2 – Inferences about Two Proportions

Objectives:
1. Find the number of successes from proportions
2. Perform calculations for testing claims
3. Construct a confidence interval estimate for the difference between two population proportions
4. Perform hypothesis tests for two proportions

Overview:
In Chapters 7 and 8 we introduced methods of inferential statistics. In Chapter 7 we presented methods of constructing confidence interval estimates of population parameters. In Chapter 8 we presented methods of testing claims made about population parameters. Chapters 7 and 8 both involved methods for dealing with a sample from a single population. The objective of this chapter is to extend the methods for estimating values of population parameters and the methods for testing hypotheses to situations involving two sets of sample data instead of just one.

The following are examples typical of those found in this chapter, which presents methods for using sample data from two populations so that inferences can be made about those populations.

- Test the claim that when college students are weighed at the beginning and end of their freshman year, the differences show a mean weight gain of 15 pounds (as in the “Freshman 15” belief).
- Test the claim that the proportion of children who contract polio is less for children given the Salk vaccine than for children given a placebo.
- Test the claim that subjects treated with Lipitor have a mean cholesterol level that is lower than the mean cholesterol level for subjects given a placebo.

Two Population Proportions:
In this section we present methods for (1) testing a claim made about the two population proportions and (2) constructing a confidence interval estimate of the difference between the two population proportions.

Notation for Two Proportions

\( p_1 = \text{population proportion} \)
\( n_1 = \text{size of the sample} \)
\( x_1 = \text{number of successes in the sample} \)
\( \hat{p}_1 = \text{the sample proportion} \)
\( \hat{q}_1 = 1 - \hat{p}_1 \)

\( p_2 = \text{population proportion} \)
\( n_2 = \text{size of the sample} \)
\( x_2 = \text{number of successes in the sample} \)
\( \hat{p}_2 = \text{the sample proportion} \)
\( \hat{q}_2 = 1 - \hat{p}_2 \)

Population 1

Population 2
Pooled Sample Proportion:
The pooled sample proportion is denoted by $\overline{p}$ and is given by:

$$\overline{p} = \frac{x_1 + x_2}{n_1 + n_2}$$

We denote the complement of $\overline{p}$ as:

$$\overline{q} = 1 - \overline{p}$$

Requirements:
1. We have proportions from two independent simple random samples.
2. For each of the two samples, the number of successes is at least 5 and the number of failures is at least 5.

Test Statistic for Two Proportions:
For $H_0: p_1 = p_2$  $H_1: p_1 \neq p_2$,  $H_1: p_1 < p_2$,  $H_1: p_1 > p_2$

$$z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\frac{\overline{p}\overline{q}}{n_1} + \frac{\overline{p}\overline{q}}{n_2}}}$$

where $p_1 - p_2 = 0$ (assumed in the Null Hypothesis).

With sample proportions:

$$\hat{p}_1 = \frac{x_1}{n_1} \quad \text{and} \quad \hat{p}_2 = \frac{x_2}{n_2}$$

and pooled sample proportion:

$$\overline{p} = \frac{x_1 + x_2}{n_1 + n_2} \quad \text{and} \quad \overline{q} = 1 - \overline{p}$$

Methods of Claim Testing:
1. P-value: Use Table A-2. (Use the computed value of the test statistic $z$ and find its $P$-value by following the procedure summarized by Figure 8-5 in the text.)

2. Critical values: Use Table A-2. (Based on the significance level $\alpha$, find critical values by using the procedures introduced in Section 8-2 in the text.)
Confidence Interval Estimate:
The confidence interval estimate of the difference \( p_1 - p_2 \) is:

\[
(\hat{p}_1 - \hat{p}_2) - E < (p_1 - p_2) < (\hat{p}_1 - \hat{p}_2) + E
\]

where the margin of error \( E \) is given by:

\[
E = z_{\alpha/2} \sqrt{\frac{\hat{p}_1 \hat{q}_1}{n_1} + \frac{\hat{p}_2 \hat{q}_2}{n_2}}
\]

Hypothesis Testing:
For tests of hypothesis made about two population proportions, we consider only tests having a null hypothesis of \( p_1 = p_2 \).

The procedure for testing a hypothesis for two population proportions using the P-value or traditional method is the same as for one population:

**General Procedure for P-Value Method:**
1. Identify the specific claim or hypothesis to be tested, and put it in symbolic form.
2. Give the symbolic form that must be true when the original claim is false.
3. Of the two symbolic expressions obtained so far, let the alternative hypothesis be the one not containing equality. Let the null hypothesis be the symbolic expression that the parameter equals the fixed value being considered.
4. Select the significance level \( \alpha \) based on the seriousness of type 1 error. Make \( \alpha \) small if the consequences of rejecting a true \( H_0 \) are severe. The values of 0.05 and 0.01 are very common.
5. Identify the statistic that is relevant to this test and determine its sampling distribution (normal or t-distribution).
6. Find the test statistic and find the P-value. Draw a graph and show the test statistic and the P-value.
7. Reject \( H_0 \) if the P-value is less than or equal to the significance level \( \alpha \). Fail to reject \( H_0 \) if the P-value is greater than \( \alpha \).
8. Restate this previous decision in simple non-technical terms and address the original claim.

**Traditional Method:**
- Find the test statistic, critical values and the critical region. Draw a graph and include the test statistic, critical values, and critical region.
- Reject \( H_0 \) if the test statistic is in the critical region. Fail to reject \( H_0 \) if the test statistic is not in the critical region.
- Restate this previous decision in simple non-technical terms and address the original claim.
**Example:** The table below lists results from a simple random sample of front-seat occupants involved in car crashes. Use a 0.05 significance level to test the claim that the fatality rate of occupants is lower for those in cars equipped with airbags.

<table>
<thead>
<tr>
<th></th>
<th>Airbag Available</th>
<th>No Airbag Available</th>
</tr>
</thead>
<tbody>
<tr>
<td>Occupant Fatalities</td>
<td>41</td>
<td>52</td>
</tr>
<tr>
<td>Total number of occupants</td>
<td>11,541</td>
<td>9,853</td>
</tr>
</tbody>
</table>

**Solution:** Requirements are satisfied: two simple random samples, the samples are independent and each has at least 5 successes and 5 failures (11,500, 41; 9801, 52).

**P-value Method:**

**Step 1:** Original claim is that the fatality rate of occupants is lower for those in cars equipped with airbags. Symbolically this is: \( p_1 < p_2 \)

**Step 2:** Opposite of original claim is \( p_1 \geq p_2 \)

**Step 3:** \( p_1 < p_2 \) does not contain equality. Therefore,

- \( H_0: p_1 = p_2 \) is the null hypothesis
- \( H_1: p_1 < p_2 \) is the alternative hypothesis.

**Step 4:** Significance level is \( \alpha = 0.05 \)

**Step 5:** The relevant statistic is the pooled sample proportion \( \bar{p} \).

\[
\bar{p} = \frac{x_1 + x_2}{n_1 + n_2}
\]

\[
\bar{p} = \frac{41 - 52}{11,541 - 9,853} = 0.004347
\]

and

\[
\bar{q} = 1 - \bar{p}
\]

\[
\bar{q} = 1 - 0.004347 = 0.995653
\]

**Step 6:** Calculate the test statistic \( z \):

\[
z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\frac{\hat{p}\hat{q}}{n_1} + \frac{\hat{p}\hat{q}}{n_2}}}
\]

\[
z = \frac{\left( \frac{41}{11,541} - \frac{52}{9,853} \right) - 0}{\sqrt{\left( 0.004347 \times 0.995653 \right) \times \left( \frac{1}{11,541} + \frac{1}{9,853} \right)}}
\]

\[
z = \frac{-0.004347}{\sqrt{0.004347 \times 0.995653 \times \left( \frac{1}{11,541} + \frac{1}{9,853} \right)}}
\]

\[
z = -1.91
\]
Left-tailed test. Area to left of $z = -1.91$ is 0.0281 (Table A-2), so the $P$-value is 0.0281.

**Step 7:** Because the $P$-value of 0.0281 is less than the significance level of $\alpha = 0.05$, we reject the null hypothesis of $p_1 = p_2$.

**Step 8:** Because we reject the null hypothesis, we conclude that there is sufficient evidence to support the claim that the proportion of accident fatalities for occupants in cars with airbags is less than the proportion of fatalities for occupants in cars without airbags. Based on these results, it appears that airbags are effective in saving lives.

**Traditional Method:**
With a significance level of $\alpha = 0.05$ in a left-tailed test based on the normal distribution, we refer to Table A-2 and find that an area of $\alpha = 0.05$ in the left tail corresponds to the critical value of $z = -1.645$. Because the test statistic falls in the critical region bounded by the critical value of $z = -1.645$ we again reject the null hypothesis.
**Example:** Use the sample data given in the preceding Example to construct a 90% confidence interval estimate of the difference between the two population proportions. (The confidence level of 90% is comparable to the significance level of $\alpha = 0.05$ used in the preceding left-tailed hypothesis test.) What does the result suggest about the effectiveness of airbags in an accident?

**Solution:** Requirements are satisfied as we saw in the preceding example.

90% confidence interval: $z_{\alpha/2} = 1.645$. Calculate the margin of error, $E$.

$$E = z_{\alpha/2} \sqrt{\frac{\hat{p}_1 \hat{q}_1}{n_1} + \frac{\hat{p}_2 \hat{q}_2}{n_2}}$$

$$E = 1.645 \sqrt{\frac{41}{11541} + \frac{52}{9853}}$$

$$E = 1.645 \sqrt{\frac{11500}{11541} + \frac{9801}{9853}}$$

$$E = 0.001507$$

Construct the Confidence Interval:

$$(\hat{p}_1 - \hat{p}_2) - E < (p_1 - p_2) < (\hat{p}_1 - \hat{p}_2) + E$$

$$(0.003553 - 0.005278) - 0.001507 < (p_1 - p_2) < (0.003553 - 0.005278) + 0.001507$$

$$-0.001725 - 0.005278 < (p_1 - p_2) < -0.001725 + 0.005278$$

$$-0.00323 < (p_1 - p_2) < 0.00218$$

The confidence interval limits do not contain 0, implying that there is a significant difference between the two proportions. The confidence interval suggests that the fatality rate is lower for occupants in cars with air bags than for occupants in cars without air bags. The confidence interval also provides an estimate of the amount of the difference between the two fatality rates.
**Example:** In a random sample of 360 women, 65% favored stricter gun control laws. In a random sample of 220 men, 60% favored stricter gun control laws. Test the claim that the proportion of women favoring stricter gun control is higher than the proportion of men favoring stricter gun control. Use a significance level of 0.05.

**Solution:** All requirements are met. Simple random samples of independent events.

**Step 1:** Original claim is that the proportion of women favoring stricter gun control is higher than the proportion of men favoring stricter gun control. \( p_1 > p_2 \)

**Step 2:** Opposite of original claim is \( p_1 \leq p_2 \)

**Step 3:** does not contain equality. Therefore,  
\( H_0: \ p_1 = p_2 \) is the null hypothesis  
\( H_1: \ p_1 > p_2 \) is the alternative hypothesis.

**Step 4:** Significance level is 0.05

**Step 5:** The relevant statistic is

\[
\hat{p}_1 = \frac{x_1}{n_1} = \frac{234}{360} = 0.65 \\
\hat{p}_2 = \frac{x_2}{n_2} = \frac{132}{220} = 0.60 \\
\bar{p} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{366}{580} = 0.63103 \\
\bar{q} = 1 - \bar{p} = 1 - 0.63103 = 0.368966
\]

**Step 6:** Calculate the test statistic \( z \):

\[
z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\frac{\bar{p}\bar{q}}{n_1} + \frac{\bar{p}\bar{q}}{n_2}}} = 1.21
\]

**Step 7:** From Table A-2, the P-value is 1-0.8869 = 0.1131. Because 0.1131>0.05, we fail to reject the null hypothesis.

**Step 8:** There is not sufficient sample evidence to support the claim that the proportion of women favoring stricter gun control is higher than the proportion of men.
**Example:** A researcher finds that of 1000 people who said that they attend a religious service at least once a week, 31 stopped to help a person with car trouble. Of 1200 people interviewed who had not attended a religious service at least once a month, 22 stopped to help a person with car trouble. At the 0.05 significance level, test the claim that the two proportions are equal.

**Solution:** All requirements are met. Simple random samples of independent events.

**Step 1:** Original claim is

**Step 2:** Opposite of original claim is

**Step 3:** does not contain equality. Therefore,

- \( H_0: \) is the null hypothesis
- \( H_1: \) is the alternative hypothesis.

**Step 4:** Significance level is

**Step 5:** The relevant statistic is

**Step 6:** Calculate the test statistic \( z \):

**Step 7:** P-value, critical values, accept or reject null hypothesis.

**Step 8:** Conclusion