Solving Radical Equations

Equations with Radicals:
A radical equation is an equation in which a variable appears in one or more radicands. Some examples of radical equations are:

\[ \sqrt{x} + 5 = 7 \quad \sqrt{2x + 7} = 3 \quad \sqrt{3x + 1} + 4 = 0 \]

Solution of a Radical Equation:
The solution of a radical equation is the value of the variable that satisfies the radical equation, that is, makes the radical equation true.

Example: Is the value \( x = 2 \) a solution to the radical equation \( \sqrt{x + 7} = 3 \)?

Solution: Substitute the value of \( x \) into the equation and evaluate.

\[ \sqrt{x + 7} = 3 \]
\[ \sqrt{2 + 7} = 3 \]
\[ \sqrt{9} = 3 \]
\[ 3 = 3 \]

Since the equation results in an identity, \( x = 2 \) is a solution to the equation.

Solving a Radical Equation:
When solving an equation containing a radical, the primary objective is to isolate the term containing the radical. Once this is accomplished, raise both sides of the equation to a power that is equal to the index of the radical in the equation.

Example: Solve the equation \( \sqrt{x} + 5 = 7 \).

Solution: Isolate the radical, and then raise both sides of the equation to the 2\(^{nd}\) power.

\[ \sqrt{x} + 5 = 7 \]
\[ \sqrt{x} = 2 \]
\[ \left( \sqrt{x} \right)^2 = 2^2 \]
\[ x = 4 \]

We can check the solution by substituting the solution into the original equation.

\[ \sqrt{x} + 5 = 7 \]
\[ \sqrt{4} + 5 = 7 \]
\[ 2 + 5 = 7 \]
\[ 7 = 7 \]
Extraneous Solutions:

When raising both sides of an equation to the same power, the resulting equation may have one or more solutions that do not satisfy the original equation, that is, extraneous solutions. So when solving a radical equation, it is very important to check all solutions in the original equation.

Example: Solve the equation $\sqrt{x} + 2 = x$

Solution: Isolate the radical and then raise both sides to the 2nd power.

\[
\begin{align*}
\sqrt{x} + 2 &= x \\
\sqrt{x} &= x - 2 \\
\left(\sqrt{x}\right)^2 &= (x - 2)^2 \\
x &= x^2 - 4x + 4 \\
x^2 - 5x + 4 &= 0 \\
(x - 4)(x - 1) &= 0 \\
x &= 4 \\
x &= 1
\end{align*}
\]

By checking both solutions in the original equation, we determine that $x = 1$ is not a solution. Therefore, the only solution is $x = 4$

Example: Solve the equation $\sqrt{x} + 12 = x$

Solution: Isolate the radical and then raise both sides to the reciprocal power.

\[
\begin{align*}
\sqrt{x} + 12 &= x \\
\sqrt{x} &= x - 12 \\
\left(\sqrt{x}\right)^2 &= (x - 12)^2 \\
x &= x^2 - 24x + 144 \\
x^2 - 25x + 144 &= 0 \\
(x - 16)(x - 9) &= 0 \\
x &= 16 \\
x &= 9
\end{align*}
\]

By checking both solutions in the original equation, we determine that $x = 9$ is not a solution. Therefore, the only solution is $x = 16$
**Example:** Solve the equation $\sqrt[3]{2x + 7} = 3$.

**Solution:** We will raise both sides of this equation to the 3rd power because the index of the radical is 3.

\[
\begin{align*}
\sqrt[3]{2x + 7} &= 3 \\
(\sqrt[3]{2x + 7})^3 &= 3^3 \\
2x + 7 &= 27 \\
x &= 10
\end{align*}
\]

Checking this solution in the original equation we may determine that $x = 10$ is a solution.

**Example:** Solve the equation $\sqrt{3x + 1} + 4 = 0$.

**Solution:** Isolate the radical and square both sides of the equation.

\[
\begin{align*}
\sqrt{3x + 1} + 4 &= 0 \\
\sqrt{3x + 1} &= -4 \\
(\sqrt{3x + 1})^2 &= (-4)^2 \\
3x + 1 &= 16 \\
x &= 5
\end{align*}
\]

By checking this solution in the original equation we may determine that $x = 5$ is not a solution. Consequently, there is no solution.

**Equations with Two Radicals:**

If an equation contains two radicals it may or may not be possible to isolate one of the radicals. If it is possible, then isolate the radical with the most complex radicand.

**Example:** Solve the equation $\sqrt{4x + 5} - \sqrt{7x - 4} = 0$.

**Solution:** In this example both radicals can be isolated by moving one to the other side of the equation.

\[
\begin{align*}
\sqrt{4x + 5} - \sqrt{7x - 4} &= 0 \\
\sqrt{4x + 5} &= \sqrt{7x - 4} \\
(\sqrt{4x + 5})^2 &= (\sqrt{7x - 4})^2 \\
4x + 5 &= 7x - 4 \\
x &= 3
\end{align*}
\]

By checking this solution in the original equation we may determine that $x = 3$ is a solution.
**Example:** Solve the equation \( \sqrt{2x+1} - \sqrt{x} = 1 \)

**Solution:** Isolate the radical and then raise both sides to the 2nd power.

\[
\sqrt{2x+1} - \sqrt{x} = 1 \\
\sqrt{2x+1} = \sqrt{x} + 1 \\
\left(\sqrt{2x+1}\right)^2 = \left(\sqrt{x} + 1\right)^2 \\
2x + 1 = x + 2\sqrt{x} + 1 \\
x = 2\sqrt{x} \\
(x)^2 = (2\sqrt{x})^2 \\
x^2 = 4x \\
x^2 - 4x = 0 \\
x(x - 4) = 0 \\
x = 0 \\
x = 4
\]

By checking both solutions in the original equation, we determine that both solutions check.

**Example:** Solve the equation \( \sqrt{3x - 2} - \sqrt{x} = 2 \)

**Solution:** Isolate the radical and then raise both sides to the 2nd power.

\[
\sqrt{3x - 2} - \sqrt{x} = 2 \\
\sqrt{3x - 2} = \sqrt{x} + 2 \\
\left(\sqrt{3x - 2}\right)^2 = \left(\sqrt{x} + 2\right)^2 \\
3x - 2 = x + 4\sqrt{x} + 4 \\
2x - 6 = 4\sqrt{x} \\
x - 3 = 2\sqrt{x} \\
(x - 3)^2 = (2\sqrt{x})^2 \\
x^2 - 6x + 9 = 4x \\
x^2 - 10x + 9 = 0 \\
(x - 9)(x - 1) = 0 \\
x = 9 \\
x = 1
\]

By checking both solutions in the original equation, we determine that \( x = 1 \) is not a solution. Therefore, the only solution is \( x = 9 \)
8.6-Applications:

Example: The diagonal of a rectangular box with a square base can be calculated using the formula 
\[ d = \sqrt{2x^2 + h^2} \] where \( x \) is the length of the side of the square base.

If the diagonal of the box is 18 in. and the side of the box is 8 in., find the height of the box.

Solution: Substitute the given values into the equation and solve for \( h \).

\[
\begin{align*}
  d &= \sqrt{2x^2 + h^2} \\
  18 &= \sqrt{2(8)^2 + h^2} \\
  18 &= \sqrt{128 + h^2} \\
  18^2 &= (\sqrt{128 + h^2})^2 \\
  324 &= 128 + h^2 \\
  196 &= h^2 \\
  h^2 - 196 &= 0 \\
  (h - 14)(h + 14) &= 0 \\
  h &= 14 \\
  h &= -14
\end{align*}
\]

Because a negative answer does not make sense in the context of this problem, the height of the box is 14 in.
Example: According to Einstein’s theory of special relativity, time will pass more quickly on Earth than it will in a spaceship travelling near the speed of light (186,000 miles/second.) The special relativity equation

\[ R_a = R_f \sqrt{1 - \left(\frac{v}{c}\right)^2} \]

gives the aging rate of an astronaut, \( R_a \), relative to the aging rate of a friend, \( R_f \), on Earth. In this formula, \( v \) is the astronaut’s speed and \( c \) is the speed of light. If the astronaut travels at 90% of the speed of light for 2 years, how many years will the astronaut’s friend on Earth have aged during this time period? When the astronaut reaches the speed of light, how much will he age relative to his friend on Earth?

Solution: To answer the first question, let \( v = 0.9c \) and \( R_a = 2 \) Substitute these values into the equation and solve for \( R_f \).

\[
2 = R_f \sqrt{1 - \left(\frac{0.9c}{c}\right)^2} \\
2 = R_f \sqrt{1 - (0.9)^2} \\
2 = R_f \sqrt{1 - 0.81} \\
2 = R_f \sqrt{0.19} \\
R_f = \frac{2}{\sqrt{0.19}} \\
R_f = 4.59
\]

During the 2 years the astronaut has been in the spaceship at 90% of the speed of light, his friend on Earth has aged approximately 4.5 years. When the astronaut reaches the speed of light, substitute \( v = c \) into the equation and solve for \( R_a \).

\[
R_a = R_f \sqrt{1 - \left(\frac{c}{c}\right)^2} \\
R_a = R_f \sqrt{1 - 1^2} \\
R_a = R_f \sqrt{0} \\
R_a = 0
\]

Consequently, Einstein’s’ theory of special relativity seems to suggest mathematically the possibility of agelessness or eternal existence.